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1987 J. Phys. A: Math. Gen. 20 L1135

(http://iopscience.iop.org/0305-4470/20/17/001)

J. Phys. A: Math. Gen. **20** (1987) L1135–L1138. Printed in the UK

LETTER TO THE EDITOR

An obstruction to generalising superalgebras

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Received 1 September 1987

Abstract. It is shown that a G-graded algebra in which xy is a scalar multiple of yx is just a twisted version of a Z_2-graded superalgebra.

Superalgebras have proved to be such a useful class of Z_2-graded algebras that it is tempting to consider analogues where Z_2 is replaced by another group. In this letter we shall show that the scope for such a generalisation is rather limited.

We first recall that an algebra $\mathscr{A}$ is said to be G-graded (with G a discrete group) if, as a vector space, it decomposes into a direct sum

$$\mathscr{A}=\bigoplus_{g\in G} \mathscr{A}_g$$

in such a way that $\mathscr{A}_g \cdot \mathscr{A}_h = \mathscr{A}_{gh}$ for all $g, h \in G$. When $\mathscr{A}$ is a Banach algebra the representation theory of such 'Banach algebra bundles' has been studied by Fell (1969).

The particular feature of superalgebras which gives them such a rich calculus is the additional assumption relating xy to yx. If $\mathscr{A}$ is a complex algebra we can easily generalise this by the assumption that

$$xy=\varepsilon(g, h)yx$$

whenever $x \in \mathscr{A}_g$ and $y \in \mathscr{A}_h$, where $\varepsilon(g, h)$ is a non-zero complex number.

Since

$$xy = \varepsilon(g, h)yx = \varepsilon(g, h)\varepsilon(h, g)xy$$

we see that unless $\mathscr{A}_g \cdot \mathscr{A}_h = 0$ we must have

$$\varepsilon(g, h)\varepsilon(h, g) = 1$$

i.e. ε is antisymmetric. (In fact, we may always take ε to be antisymmetric since when $\mathscr{A}_g\mathscr{A}_h = 0$ it may be changed with impunity.) In particular,

$$\varepsilon(g, g)^2 = 1$$

so that $\varepsilon(g, g)$ is always 1 or -1.

We note that, up until now, we have not needed to assume that $\mathscr{A}$ is associative, so that the argument would have applied equally to graded Lie algebras, but for the next step we shall assume that $\mathscr{A}$ is associative. Then for $z \in \mathscr{A}_k$

$$\varepsilon(gh, k)zxy = xyz = \varepsilon(h, k)xzy = \varepsilon(g, k)\varepsilon(h, k)zxy$$

so that unless $\mathscr{A}_g\mathscr{A}_h\mathscr{A}_k = 0$ we have

$$\varepsilon(gh, k) = \varepsilon(g, k)\varepsilon(h, k).$$

By antisymmetry ε also defines a linear character in its second variable, so that we deduce that ε is a bicharacter on G (i.e. a character in each variable). This immediately tells us that ε depends on G only through the Abelian quotient G/[G, G], (where [G, G] denotes the commutator subgroup). Consequently we may as well assume that G is Abelian.

We notice also for further use that

$$\begin{aligned}\varepsilon(gh, gh) &= \varepsilon(h, h)\varepsilon(h, g)\varepsilon(g, h)\varepsilon(g, g)\\ &= \varepsilon(h, h)\varepsilon(g, g)\end{aligned}$$

so that $g \to \varepsilon(g, g)$ defines a homomorphism from G into the two-element group $\{\pm 1\}$. We shall therefore find it useful to introduce the Z_2-valued character $\varepsilon(g) = \varepsilon(g, g)$. Examples such as the irrational rotation algebra (Rieffel 1981) (arising from an integer version of the canonical commutation relations) show that this is about as much as one can say in general.

One may, however, wonder how much the lack of commutativity could simply be due to the presence of a multiplier on the group. For suppose that σ is a multiplier on G and we define a new 'twisted' product

$$x_* y = \sigma(g, h)xy$$

for $x \in \mathcal{A}_g$, $y \in \mathcal{A}_h$, then

$$x_* y = \sigma(g, h)\varepsilon(g, h)yx = [\sigma(g, h)/\sigma(h, g)]\varepsilon(g, h)y_* x$$

so that ε has changed. The question is whether we can simplify or remove ε altogether by introducing such a new 'twisted' product.

In fact, Kleppner (1965, proposition 6.1) has shown that, by a suitable choice of σ, ε may be replaced by a character which is both symmetric and antisymmetric. We may as well assume that this has already been done, so that

$$\varepsilon(g^2, h) = \varepsilon(g, h)^2 = \varepsilon(g, h)\varepsilon(h, g) = 1.$$

This means that ε is now a Z_2-valued bicharacter which is lifted from the group G/G^2, and so without loss of generality we may assume that G is actually a 2-group.

We now choose a set of generators for G and for each pair $\{g, h\}$ of distinct generators we let $\sigma(g, h)$ and $\sigma(h, g)$ each take the value ± 1 related by the condition

$$\begin{aligned}\sigma(g, h)/\sigma(h, g) &= -\varepsilon(g, h) \qquad &&\text{if } \varepsilon(g) = \varepsilon(h) = -1\\ &= +\varepsilon(g, h) &&\text{otherwise}\end{aligned}$$

and let $\sigma(g, g) = 1$. If σ is extended to a bicharacter on G then it automatically satisfies the cocycle identity and so defines a multiplier. Twisting the old product by σ we obtain a new product such that

$$\begin{aligned}x_* y &= -y_* x \qquad &&\text{if } \varepsilon(g) = \varepsilon(h) = -1\\ &= y_* x &&\text{otherwise.}\end{aligned}$$

But now by setting

$$\mathcal{A}_\pm = \bigoplus_{\varepsilon(g) = \pm 1} \mathcal{A}_g$$

we have a graded algebra in the usual sense. Thus the only algebras in this class of generalised superalgebras are those obtained from ordinary superalgebras by twisting the product by a group multiplier. By exploiting some of Kleppner's more detailed results it is even possible to drop the condition that G be discrete and consider wider classes of topological groups.

If $\mathscr{A}$ is a homogeneous algebra then the method of Fell (1969) permit one to induce all its irreducible representations, but unfortunately few interesting graded algebras seem to be homogeneous.

Graded Lie algebras could be studied with the aid of the associative enveloping algebra but that is rather restrictive and it is more satisfactory to extend the above discussion directly to Lie algebras $\mathscr{A}$ graded by a group G in the sense of the earlier paragraphs. We have already observed that the arguments which show that $\varepsilon(g, h)\varepsilon(h, g) = 1$ and that $\varepsilon(g, g) = \pm 1$ are still valid. For graded Lie algebras we shall impose the additional requirement that $\varepsilon(e, e) = -1$ where e is the identity element of G. The associative law must be replaced by a generalised Jacobi identity which may be written as

$$x(yz) = (xy)z + \theta(g, h)y(xz)$$

for a suitable function θ on $\mathrm{G}\times\mathrm{G}$ and $x \in \mathscr{A}_g$, $y \in \mathscr{A}_h$, $z \in \mathscr{A}_k$.

Now

$$\begin{aligned}\varepsilon(k, h)x(yz) &= x(zy)\\ &= (xz)y + \theta(g, k)z(xy)\\ &= \varepsilon(gk, h)y(xz) + \theta(g, k)\varepsilon(k, gh)(xy)z.\end{aligned}$$

So unless the Lie multiplication is degenerate we must have

$$\varepsilon(k, h) = \theta(g, k)\varepsilon(k, gh)$$

$$\varepsilon(k, h)\theta(g, h) = \varepsilon(gk, h).$$

Multiplying the latter equation by $\varepsilon(h, k)$ we obtain

$$\theta(g, h) = \varepsilon(h, k)\varepsilon(gk, h)$$

and the former equation yields an equivalent identity on multiplying by $\varepsilon(gk, h)$.

Setting $k = e$ we deduce that

$$\theta(g, h) = \varepsilon(h, e)\varepsilon(g, h).$$

Thence

$$\begin{aligned}\theta(g, h)\theta(k, h) &= \varepsilon(gk, h)\varepsilon(h, k)\varepsilon(k, h)\varepsilon(h, e)\\ &= \varepsilon(gk, h)\varepsilon(h, e)\\ &= \theta(gk, h).\end{aligned}$$

Thence $\theta(\cdot, h)$ is a character for each $h \in \mathrm{G}$. In particular, if we write χ for the character $\theta(\cdot, e) = \varepsilon(\cdot, e)\varepsilon(e, e) = -\varepsilon(\cdot, e)$, then

$$\theta(g, h) = \varepsilon(g, h)\varepsilon(h, e) = -\chi(h)\varepsilon(g, h).$$

Substituting this back into the bicharacter property of θ we obtain

$$\chi(h)\varepsilon(g, h)\varepsilon(h, k) = -\varepsilon(gk, h).$$

Inverting this and exploiting the antisymmetry of ε, then

$$-\chi(h)^{-1}\varepsilon(h, g)\varepsilon(h, k) = \varepsilon(h, gk)$$

from which it immediately follows that $-\chi(h)^{-1}\varepsilon(h,\cdot)$ is also a character. Combining this with the earlier result we conclude that

$$-\chi(g)^{-1}\chi(h)\varepsilon(g,h)$$

defines an antisymmetric bicharacter, β. So

$$\varepsilon(g,h)=-\beta(g,h)\chi(gh^{-1})$$

$$\theta(g,h)=\beta(g,h)\chi(g)$$

with β an antisymmetric bicharacter and χ a character. By direct substitution one can easily check that for any β and χ these do satisfy the original constraints.

If one imposes the extra condition that

$$\varepsilon(g,e)=\varepsilon(e,g)=-1$$

for all g in G then $\chi=1$ and $\theta=\beta=-\varepsilon$ is an antisymmetric bicharacter. One can then show, as in the associative case, that a differs from an ordinary Lie superalgebra only by a twisting.

References

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